Relative velocity fluctuations in turbulent flows at moderate Reynolds number. II. Model calculation

P. Tong and W. I. Goldburg
Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, Pennsylvania 15260

(Received 10 February 1988; accepted 20 July 1988)

A model calculation is presented to study turbulence at moderate Reynolds number, Re. In conformity with recent measurements, it is proposed that in the energy cascade the fractional volume occupied by eddies of various sizes depends on Re. By introducing a Re-dependent parameter in the random beta model, it is shown that the scaling behavior of the small-velocity fluctuations at moderate Re can be characterized by a single Re-dependent scaling exponent α_0. The calculated Re dependence of α_0 is consistent with the experimental data. Corrections to this scaling are calculated using the multifractal concept. The Reynolds number dependence of other multifractal properties in turbulent flows is also calculated.

I. INTRODUCTION

In theories of turbulence one is interested in the small-scale statistics of the velocity difference, $V(R,t)$, between a pair of points, separated by a distance R, in the field of turbulent flow. Here $V(R,t)$ is defined as $V(R,t) = |\mathbf{v}(\mathbf{r},t) - \mathbf{v}(\mathbf{r}+\mathbf{R},t)|$, where $\mathbf{v}(\mathbf{r},t)$ is the local velocity of the fluid. The statistical property of $V(R,t)$ can be described by its probability distribution function $P(V,R)$. In general, $P(V,R)$ could be a complicated function, depending upon the Reynolds number, the boundary conditions, and other relevant system parameters. To characterize $P(V,R)$ one therefore needs a very large number of independent parameters, say, the successive velocity moments $\langle V(R,t)^p\rangle$.

In the classical picture of the eddy cascade, which originated with Kolmogorov[1] (hereafter K41), turbulence is viewed as a cascade of turbulent kinetic energy from large scales to small scales. Energy is fed into the turbulence at large-scale L_0, which is determined by the boundaries. The kinetic energy is continuously transferred from eddies of size $R < L_0$ to eddies of smaller size, until it dissipates when the size of eddies becomes comparable to the Kolmogorov dissipation length[1] L_d. In this theory $L_d = (\nu^3/\epsilon)^{1/4}$, where ϵ is the energy dissipation rate, and ν is the kinematic viscosity of fluid. When R is in the inertial range $L_d \ll R \ll L_0$, the energy cascade proceeds without dissipation, and $V(R,t)$ is expected to be self-similar, i.e., the statistical properties of $V(R,t)$ over varying length scales become identical under an appropriate scaling of velocities.[2]

According to the K41 theory, only two parameters, R and ϵ, are relevant to turbulence at scales in the inertial range. By a simple dimensional argument, one can show that only one scaling velocity is needed to characterize fully developed turbulence, namely, the characteristic velocity $u(R) = (\epsilon R)^{1/3}$, associated with eddies of size R. Therefore the moments of $V(R,t)$ obey a scaling law

$$\langle V(R,t)^p\rangle \sim u(R)^p \sim R^{\xi_p}, \tag{1}$$

where $\xi_p = p/3$. It is easy to show that the scaling in Eq. (1) follows if the distribution function $P(V,R)$ is a homogeneous function $Q(V/u(R))/u(R)$.[3] The self-similarity, or scaling, is generally thought to hold only at extremely large Reynolds numbers, where many important turbulence experiments have been carried out.

In recent years attention has been focused on the study of hydrodynamic systems in the opposite limit, where the control parameter slightly exceeds some critical value.[4,5] Two examples are Bénard convection and Couette flow in the vicinity of critical Rayleigh and Reynolds numbers, respectively. In spite of the relatively low level of excitation of these dynamic systems, they share certain attributes of highly turbulent fluids, including self-similarity and multifractal scaling of certain variables.[6,7] These variables are the probability that the system will visit regions in phase space in the case of a dynamic system, and the volume fraction occupied by the eddies of various sizes in fully developed turbulence. Highly turbulent systems differ from the dynamic systems in that very many degrees of freedom are excited in turbulence whereas only a few are needed to account for the chaotic behavior in the latter case. A question of obvious interest is the transition from chaos to fully developed turbulence. The experiments reported in the preceding paper[8] (referred to as I), suggest that scaling ideas can still be applied to the turbulence in the transition region, provided that one allows ξ_p to become dependent on Re in this transition region. The purpose of this paper is to present a model calculation that can explain the results we have observed.

As Landau first pointed out,[9] the K41 theory did not take the fluctuations of the local energy dissipation rate ϵ into account. Experiments[10,11] have provided strong evidence that the small-scale structures of fully developed turbulence become less and less space filling as R decreases, an effect that is called intermittency.[12–16] One simple model that includes the intermittency effect has become well known as the beta model.[2] The beta model assumes that the turbulent energy is transferred to only a fixed fraction β of the eddies of smaller size. The turbulent "active region," where the vorticity is highly localized, forms a homogeneous fractal with dimension D_0 embedded in the Euclidean space of dimension d. Here the volume contraction ratio β is assumed to be a constant in the range $0 \leqslant \beta \leqslant 1$. With this model, turbulence possesses global scaling character, and the

 0031-9171/88/113253-07$01.90

moments of $V(R,t)$ have the same form as that in Eq. (1) with

$$\xi_p = [\tfrac{1}{3} - (d - D_0)/3]p + d - D_0. \tag{2}$$

The second term, $-(d - D_0)p/3$, in ξ_p arises because of the increasing concentration of energy into smaller regions, which reduces the rate of variation of $\langle V(R,t)^p \rangle$ with R. The third term, $d - D_0$, comes from spatially averaging relative velocity over the turbulent active region, because the probability for being in the active region is proportional to $R^{d - D_0}$. When $D_0 = d$ the above equation reduces to that of the K41 theory. Note that both the beta model and the K41 theory predict that ξ_p is a linear function of p.

The recent experiment by Anselmet *et al.*[17] indicates that the exponent ξ_p has a weaker than linear dependence on p when $p \gtrsim 8$. To explain this nonlinear p dependence of ξ_p within the frame of the scaling theories discussed above, Frisch and Parisi introduced the multifractal model.[6] In this model the relative velocity, in a domain of size R, possesses a local scaling $V(R) \sim R^{\alpha}$, i.e., α varies from point to point in the flow. Using multifractal ideas, Benzi *et al.*[18] presented an interesting probabilistic extension of the beta model to simulate multifractal sets. The model is called the random beta model. This model assumes that the volume contraction ratio β in the original beta model is a random variable. To get a simple physical picture, it is assumed in this model that the probability distribution of β is a bimodal function with a relative probability amplitude being an adjustable parameter. We will show in this paper that the random beta model fits our measured velocity fluctuations in the intermediate range of Reynolds number, provided that one introduces a Re dependence for the probability amplitude x in the model.

The experiment described in I gives evidence that the scaling picture can even be applied to velocity fluctuations in flows at relatively low turbulence levels. We found that when Re exceeds a transition Reynolds number Re_c ($\text{Re}_c \sim 300$–400 for grid flow, and $\text{Re}_c \sim 3000$–4000 for pipe flow), the distribution function $P(V,R)$ has a scaling form $Q(V/u(R))/u(R)$, and the scaling velocity $u(R)$ has the form $u(R) \sim R^{\zeta}$. In contrast to fully developed turbulence, where ζ is a constant (close to $\frac{1}{3}$), the measured exponent ζ for turbulence at moderate Re shows a nontrivial Re dependence. Here we refer to turbulence in the range $\text{Re}_c \lesssim \text{Re} \lesssim 30\,\text{Re}_c$ as "moderate" in contrast to fully developed turbulence in the limit $\text{Re} \to \infty$. Just above Re_c, ζ has the approximate form $\zeta \sim [(\text{Re} - \text{Re}_c)/\text{Re}_c]^{\phi}$, where $\phi \sim 0.5$. Near the maximum attainable values of Re, ζ has climbed to, and saturated at, a value close to $\frac{1}{3}$, the Kolmogorov value. This crossover behavior was found both in pipe flow and in grid flow.

In conformity with our experiments mentioned above, we conjecture that the crossover behavior or the correction to scaling for turbulence at moderate Reynolds number may have a universal character. By introducing a Re-dependent probability amplitude x in the random beta model, we show that when Re is in the intermediate region, $P(V,R)$ deviates from scaling and ξ_p in Eq. (1) no longer is proportional to p. Using multifractal ideas we also show that for turbulence at moderate Re, ξ_p becomes a function of x, where x itself is a function of the reduced Reynolds number $\omega = (\text{Re} - \text{Re}_c)/\text{Re}_c$. Specifically, it will be shown that

$$\xi_p(x) = d - D_0(x) + \alpha_0(x)p + [a(x)/2]p^2 + \cdots. \tag{3}$$

In the above, $D_0(x)$ and $\alpha_0(x)$ can be thought as an "average" fractal dimension and "average" scaling exponent, respectively, both of them depending on Re. It will be shown in Sec. IV that if one keeps the linear term in Eq. (3) for small-velocity fluctuations (small p), the corresponding distribution function $P(V,R)$ is a homogeneous function of the form $P(V,R) \sim R^{d - D_0(x)} Q(V/u(R))/u(R)$. The scaling velocity $u(R)$ has the form $u(R) \sim R^{\alpha_0(x)}$. The calculated Re dependence of the scaling exponent $\alpha_0(x)$ is consistent with the experimental data in I. The measurements in I suggest that a homogeneous fractal with a Re-dependent fractal dimension $D_0(x)$ is adequate to characterize the scaling behavior of the small-velocity fluctuations at moderate Re. Corrections to this scaling behavior due to the quadratic term $[a(x)/2]p^2$ in Eq. (3) are examined. The Reynolds number dependence of other multifractal properties in turbulent flows is also calculated.

The concept of multifractal sets[6,7] and the random beta model[18] are reviewed in Sec. II. In Sec. III we extend the random beta model to allow x to depend on the Reynolds number for turbulence at moderate Re. A comparison between the experimental results in I and the model calculation in Sec. III is presented in Sec. IV. Finally, the work is summarized in Sec. V.

II. MULTIFRACTAL SETS AND THE RANDOM BETA MODEL

The notion of multifractal sets can be understood in the following way. Let us divide the turbulent flow into N small boxes of size R. Within each box one can measure the velocity difference $V(R)$ as a function of R, and a local scaling law, $V(R) \sim R^{\alpha}$, may be found for small values of R. Hence in each box

$$\lim_{R \to 0} [V(R)/R^{\alpha}] = \text{const} \quad (\neq 0). \tag{4}$$

Since both the K41 theory and the beta model ensure that $\alpha \leqslant \frac{1}{3}$, Eq. (4) indicates that the velocity gradient is a singular quantity. Thus the relative velocity $V(R)$ is said to have a singularity of the order α ($\alpha > 0$). In general, α may take on a range of values, and the distribution of α among the N boxes is nontrivial. One way to describe the set of N singular points is to partition them into subsets $S(\alpha)$, where $S(\alpha)$ is defined as a group of points for which the relative velocity $V(R)$ has a singularity of the order between α and $\alpha + d\alpha$. The subset $S(\alpha)$ is a fractal object characterized by its fractal dimension $f(\alpha)$.

With the multifractal sets $S(\alpha)$, the moments of the relative velocity $V(R,t)$ can be written as an average over all possible values of α:

$$\langle V(R,t)^p \rangle \sim \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha\, \rho(\alpha,R) R^{\alpha p}, \tag{5}$$

where $\rho(\alpha,R)$ is the probability that $V(R,t)$ has a singularity of the order of α. The spatial average over the N boxes has thus been converted to an average over α. Because $S(\alpha)$ is a fractal set with fractal dimension $f(\alpha)$ embedded in Euclidean space of dimension d ($d = 3$), the probability that a singular point belongs to $S(\alpha)$ is proportional to $R^{d-f(\alpha)}$, so we have[7]

$$\langle V(R,t)^p \rangle \sim \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha\, \rho'(\alpha) R^{\alpha p + d - f(\alpha)}, \tag{6}$$

where $\rho'(\alpha)$ is the proportionality factor in $\rho(\alpha,R)$. Because R is very small, the integral in Eq. (6) is dominated by only one value of α, namely, the one that makes the exponent $\alpha p + d - f(\alpha)$ smallest, provided that $\rho'(\alpha) \neq 0$. Using the saddle-point method, the integral in Eq. (6) can be carried out and the final result is

$$\langle V(R,t)^p \rangle \sim R^{\xi_p}, \tag{7}$$

where

$$\xi_p = \alpha p + d - f(\alpha). \tag{8}$$

The exponent α must satisfy the condition

$$\frac{d}{d\alpha}[\alpha p + d - f(\alpha)] = 0. \tag{9}$$

It is also assumed that

$$\frac{d^2}{d\alpha^2}[\alpha p + d - f(\alpha)] = -f''(\alpha) > 0. \tag{10}$$

Since ξ_p is the Legendre transform[19] of $d - f(\alpha)$, the inverse Legendre transformation gives

$$f(\alpha) = \alpha p - \xi_p + d, \tag{11}$$

and

$$\alpha(p) = \frac{d}{dp}\xi_p. \tag{12}$$

The exponent ξ_p in Eqs. (7) and (8) characterizes the scaling feature of turbulence. Physically, the pth moment of $V(R,t)$ picks out a particular kind of singularity in sets $S(\alpha)$, because ξ_p depends upon a specific value of $\alpha(p)$ for a given p. Equivalently, the scaling property of $V(R,t)$ may also be characterized by a continuous spectrum of the local scaling exponents $\alpha(p)$ and their densities (or fractal dimensions) $f(\alpha)$.

As mentioned in Sec. I, the random beta model proposed by Benzi *et al.*[18] gives a simple realization of multifractal sets. This model assumes that the volume contraction ratio β is a random variable with a probability distribution $\Phi(\beta)$. Using this model Benzi *et al.* have shown that the fractal dimension D_0 of the turbulent active region is

$$D_0 = d + \log_2\langle \beta \rangle, \tag{13}$$

and that ξ_p in Eq. (7) has the form

$$\xi_p = p/3 - \log_2\langle \beta^{(1-p/3)} \rangle, \tag{14}$$

where the angle brackets represent an average over β. It is clear that complete information about the probability distribution function $\Phi(\beta)$ can be obtained from a knowledge of all the moments of β (or ξ_p), and that the fractal dimension D_0 only provides "mean value"-like information.

To compare with the measurements of ξ_p, Benzi *et al.* proposed a simple form for the distribution function $\Phi(\beta)$,

$$\Phi(\beta) = (1-x)\delta(\beta - \tfrac{1}{2}) + x\delta(\beta - 1), \tag{15}$$

where x is a fitting parameter in the range $0 \leqslant x \leqslant 1$. The calculated ξ_p by this distribution fits the experimental data[17] very well when $x = 0.875$. Of course, it is purely an assumption that $\Phi(\beta)$ is a bimodal function. Benzi *et al.* have given a plausible argument[18] for this assumption.

III. AN EXTENSION OF THE RANDOM BETA MODEL

In accordance with the experimental results in I we conjecture that turbulence at moderate Re may possess some universal properties. Presumably, if the scaling picture is a good approximation to the turbulence at moderate Re, only a few control parameters are needed to determine $f(\alpha)$ or ξ_p. Clearly, the Reynolds number is a relevant parameter for the turbulence in this region. A simple way to introduce a Re-dependent ξ_p or $f(\alpha)$ is to let the probability amplitude x in Eq. (15) be a function of Re rather than a constant. We propose that $x = x(\omega)$, where $\omega = (\mathrm{Re} - \mathrm{Re}_c)/\mathrm{Re}_c$, and that $x(\omega)$ is an increasing function of ω. This picture that for low Reynolds number turbulence "vortex sheets" ($\beta = \frac{1}{2}$) are the dominant structure, is consistent with the recent computer simulation of the forced Navier–Stokes equation by Kerr.[20]

From Eqs. (14) and (15) we have

$$\xi_p(x) = p/3 - \log_2[(1-x)2^{p/3-1} + x]. \tag{16}$$

Figure 1 displays plots of $\xi_p(x)$ vs p for different values of x. It is seen that $\xi_p(x)$ is a linear function of p when $x = 1$ (the Kolmogorov case), and the interval in p ($p > 0$), over which $\xi_p(x)$ varies linearly with p, decreases as x decreases below unity. Finally when $x = 0$, $\xi_p(x)$ becomes a constant.

Using Eqs. (12) and (16) we obtain the local scaling exponent $\alpha(p,x)$ as

$$\alpha(p,x) = \tfrac{1}{3}(1 - \{(1-x)/[1 + x(2^{1-p/3} - 1)]\}). \tag{17}$$

Figure 2 shows how $\alpha(p,x)$ varies with p for three values of x. It is clearly shown that $\alpha(p,x)$ as a function of p decreases sharply from $\frac{1}{3}$ to 0, with the inflection point shifting to increasing p as x is increased.

With Eqs. (16), (11), and (17) the function $f(\alpha,x)$ can be calculated. First, by solving Eq. (17) for p in terms of α, one can obtain $p(\alpha,x)$, which satisfies Eq. (12). Substituting this $p(\alpha,x)$ and Eq. (16) into Eq. (11) we have, for $d = 3$,

$$f(\alpha,x) = 3 - \log_2\left[\left(\frac{3\alpha}{x}\right)\left(\frac{2x(\frac{1}{3} - \alpha)}{\alpha(1-x)}\right)^{1-3\alpha}\right]. \tag{18}$$

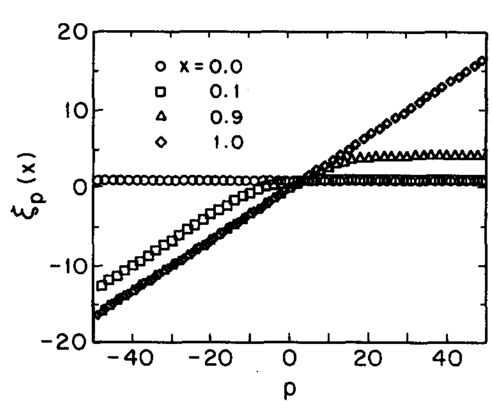

FIG. 1. The function $\xi_p(x)$ vs p at four values of x [cf. Eq. (16)].

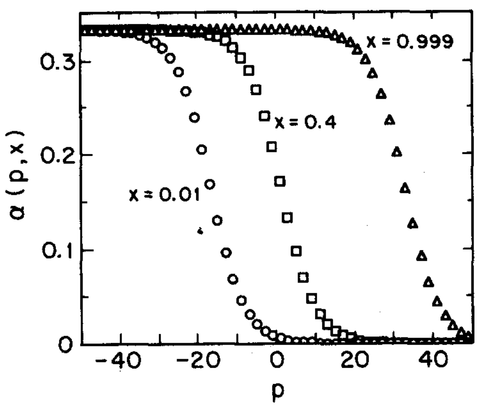

FIG. 2. The variation of the local scaling exponent $\alpha(p,x)$ with p at three values of x [cf. Eq. (17)].

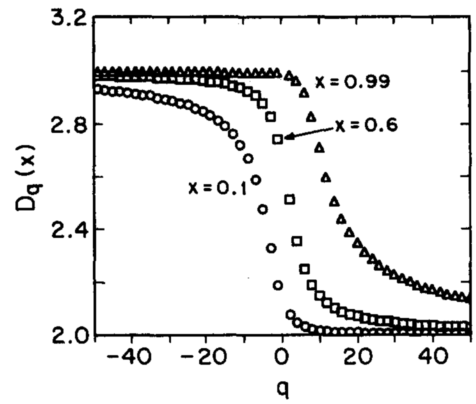

FIG. 4. The generalized dimension $D_q(x)$ vs q at indicated values of x [cf. Eq. (20)].

Figure 3 shows $f(\alpha,x)$ vs α for various values of x. Two trivial cases appear, namely, when $x=0$, the local scaling exponent $\alpha(p,0)=0$ and the corresponding fractal dimension $f(0,0)=2$. Similarly when $x=1$, $\alpha(p,1)=\frac{1}{3}$ and $f(\frac{1}{3},1)=3$. The position $\alpha_0(x)=\alpha(0,x)$, at which $f(\alpha,x)$ has its maximum value $f(\alpha_0,x)$, is seen to increase continuously from 0 to $\frac{1}{3}$ as x is increased from 0 to 1. Similarly the fractal dimension $f(\alpha_0,x)$ increases from 2 to 3.

The generalized dimension D_q, introduced by Hentschel *et al.*,[21,22] is related to $\xi_p(x)$ by the following equation[23,24]:

$$\xi_p(x) = (p/3-1)(D_{p/3}+1-d)+1. \tag{19}$$

Eliminating $\xi_p(x)$ from Eqs. (19) and (16), and solving for D_q gives

$$D_q(x) = -(q-1)^{-1}\log_2[1+x(2^{1-q}-1)]+d-1, \tag{20}$$

where $q=p/3$. Note that $D_{q=0}(x)$, which is the same as that in Eq. (13), equals $f(\alpha_0,x)$ in Eq. (18). Plots of $D_q(x)$ vs q for three values of x are displayed in Fig. 4.

Comparing the generalized dimension D_q in Fig. 4 with those obtained in some other models, such as the two-scale Cantor set, circle maps, etc.,[7] we find that the general functional form of D_q for different systems is similar, namely, D_q changes its value dramatically only in a small range of q. Below and above this range, D_q as a function of q is roughly a constant, characterized by its two asymptotic values, $D_{-\infty}$ and D_∞, respectively. Hereafter we will call this small range of q, over which D_q changes rapidly, as the transition range of D_q. Well inside the transition range D_q can be approximated by a linear function of q, which is the lognormal approximation[24] (see Sec. IV). This character of D_q [or $\alpha(p)$] is general for all the multifractal models with two competing factors, though the values of the parameters $D_{-\infty}$ and D_∞ are system dependent. Therefore we conclude that the essence of the multifractal model is to introduce a "cutoff" for D_q [or $\alpha(p)$] from being $D_{-\infty}$ to D_∞. In other words, two different scaling behaviors can be seen when q is below and above the transition region, which are characterized by $D_{-\infty}$ and D_∞, respectively. The crossover between the two scalings can be approximately described by a lognormal distribution. A nonfractal model by Nakano and Nelkin[25,26] gives a similar argument. The new feature of our model is that the transition range of $D_q(x)$ depends on the Reynolds number.

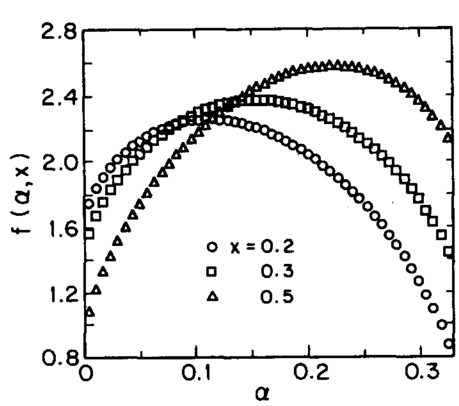

FIG. 3. The function $f(\alpha,x)$ vs α at indicated values of x [cf. Eq. (18)].

It will be shown in Sec. IV that the measured correlation function, $g(t)$, of the light intensity scattered by small particles suspended in the turbulent fluid only senses the characteristic velocity, $u(R)$, of the velocity fluctuations, i.e., it only picks up singularities with small p. To make contact with the experimental results in I, we concentrate on the small p ($p\sim1$) behavior of $\xi_p(x)$. Using Eqs. (16), (17), and (20), we get the Taylor expansion of $\xi_p(x)$ around $p=0$ up to the second order:

$$\xi_p(x) = [a(x)/2]p^2+\alpha_0(x)p+d-D_0(x), \tag{21}$$

where

$$D_0(x) = f(\alpha_0,x) = 2+\log_2(1+x),$$
$$\alpha_0(x) = \alpha(0,x) = 2x/[3(1+x)],$$
$$a(x) = -2x(1-x)\ln 2/[9(1+x)^2].$$

Note that $a(x)\leqslant 0$, and the maximum value of $|a(x)|$ is about 0.019 at $x=\frac{1}{3}$.

It should be pointed out that Eq. (21) is similar to that in the Kolmogorov lognormal model[12] except that here the three coefficients are Re dependent. In the limit $\alpha_0(x)\gg|a(x)|/2$, the quadratic term in $\xi_p(x)$ can be neglected for small p. Thus $\xi_p(x)$ becomes a linear function of p,

$$\xi_p(x) = \alpha_0(x)p + d - D_0(x). \tag{22}$$

This result is similar to the beta model result, as shown in Eq. (2), but again the coefficients $\alpha_0(x)$ and $D_0(x)$ are Re dependent.

IV. EXPERIMENTAL EVIDENCE

It was shown that the intensity correlation function, $g(t) = \langle I(0)I(t)\rangle/\langle I(0)\rangle^2$, for light scattered by small particles suspended in turbulent fluid has the form[27,28]

$$g(t) = 1 + G(kt,L), \tag{23}$$

where

$$G(kt,L) = \int_0^L dR\, h(R) \int_{-\infty}^{\infty} dV P(V,R)\cos(ktV). \tag{24}$$

In the above, $h(R) = 2(1 - R/L)/L$ is the number fraction of particle pairs separated by a distance R in the scattering volume, and V is the component of $\mathbf{V}(R,t)$ along the scattering vector $\mathbf{k}$. The scattering volume viewed by a photodetector is assumed to be quasi-one-dimensional with length L. Equations (23) and (24) tell us that the light scattered by each pair of particles contributes a phase factor $\cos(ktV)$ (because of frequency beating) to the intensity correlation function $g(t)$, and $g(t)$ is an incoherent sum of these ensemble averaged (or time averaged) phase factors over all the particle pairs in the scattering volume.

To compare with the experimental results, the distribution function $P(V,R)$ must be obtained first from the knowledge of the velocity moments calculated in Sec. III. As shown in Eq. (22), in the limit $\alpha_0(x) \gg |a(x)|/2$, $\xi_p(x)$ is a linear function of p. It is easy to show that the following form of $P(V,R)$ can give the result shown in Eq. (22):

$$P(V,R) = (R/L_0)^{d - D_0(x)} Q(V/u(R))/u(R), \tag{25}$$

where the scaling velocity $u(R) \sim R^{\alpha_0(x)}$, and L_0 is the size at which the energy is injected into the turbulent flow. Clearly, $(R/L_0)^{d - D_0(x)}$ is the probability that a pair of points in the flow, separated by a distance R, belongs to the turbulent active region of dimension $D_0(x)$, and the homogeneous function $Q(V/u(R))/u(R)$ is the velocity distribution function in the active region of turbulence. This suggests that a fractal with a Re-dependent fractal dimension $D_0(x)$ is adequate to characterize the scaling behavior of the small-velocity fluctuations when $\mathrm{Re} \gtrsim \mathrm{Re}_c$.

Quite generally, a homogeneous distribution function of the form of Eq. (25) corresponds to an exponent $\xi_p(x)$, which is proportional to p. In the K41 theory the scaling velocity $u(R)$ in Eq. (25) equals $(\epsilon R)^{1/3}$, which is the case when $x = 1$ in our model. In the beta model,[2] $u(R) = (\epsilon R)^{1/3}(R/L_0)^{-(d - D_0)/3}$, and D_0 is a constant. With the distribution function, Eq. (25), Eq. (24) becomes

$$G(kt,L) = \int_0^L dR\, h(R)\left(\frac{R}{L_0}\right)^{d - D_0(x)} F(ku(R)t), \tag{26}$$

where $F(ku(R)t)$ is the Fourier cosine transform of $Q(V/u(R))$. Clearly, $h(R)(R/L_0)^{d - D_0(x)}$ can be thought of as a joint probability of finding a pair of particles separated by a distance R in the active region of turbulent flow.

Measurements of the intensity correlation function $g(t)$ in I and in our other experiments[28,27] suggest that for small-velocity fluctuations, the characteristic function F has exponential form. If we assume $F(ku(R)t) \sim \exp[-ku(R)t]$, where $u(R) \sim R^{\alpha_0(x)}$, Eq. (26) becomes

$$G(kt,L) = 2\left(\frac{L}{L_0}\right)^{3\gamma} \exp(-\kappa) \sum_{n=0}^{\infty} \kappa^n \times \left(\frac{\Gamma\{(3+9\gamma)/(1-3\gamma)+1\}}{(1+3\gamma)\Gamma\{(3+9\gamma)/(1-3\gamma)+1+n\}} - \frac{\Gamma\{(6+9\gamma)/(1-3\gamma)+1\}}{(2+3\gamma)\Gamma\{(6+9\gamma)/(1-3\gamma)+1+n\}}\right). \tag{27}$$

Here $\gamma = [d - D_0(x)]/3$, and $\Gamma\{z\}$ is the gamma function. In the above, $\kappa = ktu(L) \sim ktL^{\alpha_0(x)}$. The measured $G(kt,L)$ fits Eq. (27) very well.[27] From Eq. (27) it is apparent that $G(kt,L)$ is a homogeneous function $G(\kappa)$, and that for short times t, $\log G(kt,L)$ is a linear function of t with a slope proportional to $ku(L)$. The reciprocal of the slope gives the characteristic decay time of $G(kt,L)$, $T = [ku(L)]^{-1} \sim L^{-\alpha_0(x)}$. Comparing this result with the measured decay time, $T \sim L^{-\zeta}$, in Fig. 7 of I, we see that $\zeta = \alpha_0(x)$.

If we include the effect of $a(x)$, the exponent $\xi_p(x)$ becomes a parabolic function of p as shown in Eq. (21). A specific distribution function, which can produce the result of Eq. (21), is the lognormal distribution,

$$P(V,R) = (R/L_0)^{d - D_0(x)} Q(V,R), \tag{28}$$

where

$$Q(V,R) = (1/\sqrt{2\pi}\sigma V)\exp[-(\ln V - \mu)^2/2\sigma^2]. \tag{29}$$

In the above $\mu = \ln[V_0(R/L_0)^{\alpha_0(x)}]$, and $\sigma^2 = a(x)\ln(R/L_0)$, where V_0 has the dimension of the relative velocity.

With the distribution of Eqs. (28) and (29), $G(kt,L)$ in Eq. (24) can be calculated. First, by changing the integration variable, the ensemble average of $\cos(kVt)$ in Eq. (24) can be carried out, and the result is

$$\int_{-\infty}^{\infty} dV Q(V,R)\cos(kVt) = \sum_{n=0}^{\infty} \frac{(-)^n K^{2n}}{(2n)!}\left(\frac{R}{L_0}\right)^{2a(x)n^2}, \tag{30}$$

where $K = V_0 kt(R/L_0)^{\alpha_0(x)}$. Inserting Eq. (30) into Eq. (24), and only keeping the first two terms for the small t limit, we obtain

$$G(kt,L) \simeq A\{1 - B[ktV_0(L/L_0)^{\alpha_0(x) + a(x)}]^2\}, \tag{31}$$

where A and B are two irrelevant coefficients. Thus the short-time behavior of $G(kt,L)$ can be characterized by a decay time $T(L) \sim L^{-\alpha_0(x) - a(x)}$. According to this equation, the measured exponent ζ in I has the form

$$\zeta = \alpha_0(x) + a(x). \tag{32}$$

To have a numerical comparison between our calculations and the experimental results in I, the functional form of $x(\mathrm{Re})$ must be given in advance. A simple function that gives a good fit to our measurement of $\zeta(\mathrm{Re})$ is

$$x(\omega) = A \tanh(C\omega^{\phi}). \quad (33)$$

Here $\omega = (\mathrm{Re} - \mathrm{Re}_c)/\mathrm{Re}_c$, and C and ϕ are parameters chosen to fit the experimental data in I. The parameter A is the asymptotic value of $x(\omega)$ when $\mathrm{Re} \to \infty$. If the turbulence is space filling and fully developed (Re very large) then $A = 1$. To include the effect of intermittency, we set $A < 1$. For example, Benzi *et al.*[18] analyzed the experimental data of Anselmet *et al.*[17] and found $A = 0.875$, according to their model. Because of the relatively large experimental uncertainty in $\zeta(\mathrm{Re})$, both values of A can fit our data.

The transition Reynolds number Re_c in the definition of ω is that value of Re below which $G(kt,L)$ no longer has scaling form and where ζ approaches zero from above (see I). We calculate $\alpha_0[x(\omega)]$ as a function of ω explicitly by using Eqs. (33) and (17). Figures 5 and 6 show our meaurements of ζ vs ω in the turbulent grid flow and pipe flow, respectively. The data are taken from Fig. 7 of I. Also shown is the calculated function $\alpha_0(\omega)$ (solid curve). The fitting parameters in the plots are $C = 0.5 \pm 0.1$, $\phi = 0.5 \pm 0.1$ for the grid flow, and $C = 0.3 \pm 0.1$, $\phi = 0.6 \pm 0.15$ for the pipe flow. Using Eqs. (33) and (17) we have, for small ω,

$$\alpha_0(\omega) \sim \omega^{\phi}, \quad (34)$$

which is the fitting function we used in Fig. 8 of I. For the grid flow the fitted value of ϕ as shown above is consistent with that in Fig. 8 of I. For the pipe flow the fitted value of ϕ is twice as large as that in Fig. 8 of I. In Fig. 8 of I we fit the data in pipe flow out to $\omega \simeq 10$, and here we fit the data out to $\omega \simeq 27$. This large uncertainty in ϕ is because we only have a few data points for $\zeta(\omega)$ at small ω in pipe flow (ϕ is sensitive to the data at small ω). Drawing a parallel with the mean field theory of phase transitions, Eqs. (33) and (34) are suggestive of the development of a local order in strong turbulence.

The effect of $a(x)$ on ζ in Eq. (32) is small. Figure 5 displays the difference of the two curves: $\alpha_0(\omega)$ vs ω (solid line) and $\alpha_0(\omega) + a(\omega)$ vs ω (dashed line). The relative difference between the two curves, $|a(x)|/\alpha_0(x)$, increases as Re is decreased but is no more than 23% when $\mathrm{Re} \gtrsim \mathrm{Re}_c$. The fitting of $\alpha_0(\omega) + a(\omega)$ to the data is slightly better than that of $\alpha_0(\omega)$. From Eq. (3) it is apparent that the deviation of $\xi_p(x)$ from linear p dependence of Eq. (22), or $P(V,R)$ from the scaling form of Eq. (25), will increase for the large-velocity fluctuations (large p) at fixed Re. Our measurements in I were not sensitive enough to establish the presence of corrections to $\xi_p(x)$ from higher powers of p in Eq. (3). Put in another way, the data were not sufficiently accurate to determine whether the low Reynolds number turbulence should be described by the multifractal model presented here, or merely by a homogeneous fractal with a fractal dimension that increases with Reynolds number. Further measurements for higher moments of velocity fluctuations are required in order to verify the present calculation.

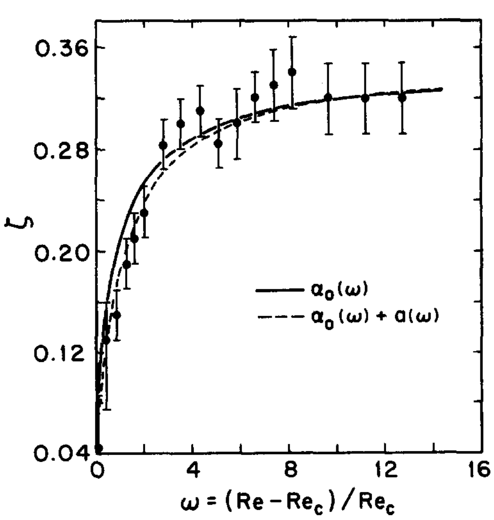

FIG. 5. The measured scaling exponent ζ vs $\omega = (\mathrm{Re} - \mathrm{Re}_c)/\mathrm{Re}_c$ in turbulent grid flow. The solid curve is a fit to $\alpha_0(\omega)$, and the dashed curve is a fit to $\alpha_0(\omega) + a(\omega)$ with $C = 0.6$ and $\phi = 0.5$ [cf. Eqs. (32) and (33)].

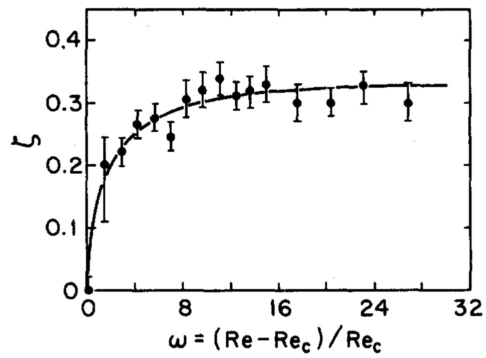

FIG. 6. The measured scaling exponent ζ vs $\omega = (\mathrm{Re} - \mathrm{Re}_c)/\mathrm{Re}_c$ in turbulent pipe flow. The solid curve is a fit to $\alpha_0(\omega)$.

V. CONCLUSION

In paper I and the present paper we have investigated turbulent flows at moderate Reynolds numbers. The experiment[8] described in I suggests that the scaling picture can be applied to the small-velocity fluctuations in turbulent flows at moderate Re, provided that one introduces a Re-dependent scaling parameter. In particular, we find that when $\mathrm{Re} > \mathrm{Re}_c$ the distribution function $P(V,R)$ has a scaling form $Q[V/u(R)]/u(R)$, where the characteristic velocity $u(R)$ obeys the scaling law, $u(R) \sim R^{\zeta}$. In contrast to fully developed turbulence, where ζ is a constant, the measured exponent ζ for turbulence at moderate Re shows a nontrivial Re dependence. In the vicinity of Re_c, ζ as a function of Re is approximately of the form $\zeta \sim [(\mathrm{Re} - \mathrm{Re}_c)/\mathrm{Re}_c]^{\phi}$, where $\phi \sim 0.5$. Near the maximum attainable values of Re, ζ has climbed to, and saturated at, a value close to $\frac{1}{3}$, the Kolmogorov value. This transition behavior is found both in pipe flow and in grid flow.

In conformity with the above experimental results, we conjecture that turbulence in the transition region can be characterized by a general scaling theory. In this paper a model calculation is presented to study the scaling behavior of turbulence at moderate Re. In particular, we propose that the fractional volume occupied by eddies of various sizes depends on the Reynolds number. To achieve this, we extend the random beta model by introducing a Re-dependent probability amplitude $x(\omega)$ in the bimodal distribution of β, where $\omega = (\mathrm{Re} - \mathrm{Re}_c)/\mathrm{Re}_c$ is the reduced Reynolds number. With this extension we have shown that the small-velocity fluctuations in the flows at relatively low turbulence levels can be characterized by a scaling velocity $u(R)$. This scaling velocity has the form, $u(R) \sim R^{\alpha_0(x)}$. Therefore the distribution function $P(V,R)$ can be approximated by a homogeneous function of the form, $P(V,R) \sim R^{d-D_0(x)} Q(V/u(R))/u(R)$. In the above, $D_0(x)$ and $\alpha_0(x)$ can be thought of as an "average" fractal dimension and an "average" scaling exponent, respectively, both of them depending on Re. The calculated Re dependence of the scaling exponent $\alpha_0(x)$ is consistent with the experimental data in I.

We have also calculated a correction to scaling as shown in Eq. (3), even though the measurements in I were not able to confirm the presence of this correction. It is interesting to note that in this calculation the quadratic term correction in Eq. (3), $|a(x)|/\alpha_0(x)$, increases as Re decreases toward the transition Reynolds number Re_c, but never exceeds 23% as long as $\mathrm{Re} \gtrsim \mathrm{Re}_c$. Using multifractal ideas the Re dependence of the generalized dimension $D_q(x)$ and the "spectrum of singularities" $f(\alpha,x)$ have been calculated. This calculation gives predictions for the large-velocity fluctuations in turbulent flow. These predictions should be tested by further measurements on the higher moments of the velocity fluctuations, $\langle V(R,t)^p \rangle$, or the distribution function, $P(V,R)$, of the large-velocity fluctuations at various Reynolds numbers.

ACKNOWLEDGMENTS

We are indebted to K. R. Sreenivasan for his valuable comments, and have benefited from illuminating discussions and correspondence with A. Onuki, M. Nelkin, and A. Hernandez-Machado. One of the authors (PT) would like to thank P.-Y. Lai for fruitful discussions.

This work is supported by the National Science Foundation under Grant No. DMR-8611666.

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